

Kowalevski top revisited

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Abstract

We review the separation of variables for the Kowalevski top and for its generalization to the algebra $\mathfrak{o}(4)$. We notice that the corresponding separation equations allow an interpretation of the Kowalevski top as a $B_2^{(1)}$ integrable lattice. Consequently, we apply the quadratic r -matrix formalism to construct a new 2×2 Lax matrix for the top, which is responsible for its separation of variables.

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Introduction

In 1889 Sophie Kowalevski [21] found and integrated new integrable case of rotation of a heavy rigid body around a fixed point which since then carried her name, *the Kowalevski top* (KT). In modern terms, this is an integrable system on the $\mathfrak{e}(3)$ algebra with a quadratic and a quartic (in angular momenta) integrals of motion.

Her original transformation to new variables led to the solution in terms of quadratures and, eventually, to the separation of variables for this complicated top [41, 22, 18]. No separation which is alternative to her original separation of variables is known for this system at the moment, even though there is a large body of literature dedicated to the problem (see, for instance, [9, 5, 7, 10, 6, 28] to name just a few).

The aim of the present work is to design and construct a new Lax matrix for the Kowalevski top which implies Kowalevski's separation of variables and which has a proper algebraic (r -matrix) structure. The principal existence of such (2×2) Lax matrix was predicted in author's MSc dissertation in 1985 [22] (see also [18]), after he rewrote the formula defining the separation variables for the top given in [41], and this new formula took a form of the spectral curve of the future Lax matrix. In what follows I shall explain how this Lax matrix can be reconstructed in somewhat regular way, without much of a guesswork.

In 1981 the Kowalevski top was generalized to the case of $\mathfrak{o}(4)$ algebra [14]. In 1990 for this $\mathfrak{o}(4)$ KT the separation of variables was found in [23, 19] which generalized the one for the $\mathfrak{e}(3)$ KT. In the present paper we will always consider this one-parameter, $\mathfrak{o}(4)$ extension of the Kowalevski top. We refer the reader to the work [17] for the survey of the Kowalevski top, its generalizations and other related tops.

The structure of the paper is following. In Section 1 we recall the integration in quadratures of the $\mathfrak{o}(4)$ Kowalevski top. In Sections 2 and 3 we describe its separation of variables and, in particular, write down the separation equations in the form which is best for deducing the ansatz for the future Lax matrix. As a result, we conclude that, as it follows from the separation equations, the Kowalevski top (together with its $\mathfrak{o}(4)$ version) is an integrable system with boundary conditions of $B_2^{(1)}$ -type which means that the corresponding Lax matrix should factorize and factors obey the reflection equation (quadratic r -matrix algebra). In Section 4 we introduce the corresponding quadratic algebra \mathcal{B} . Section 5 contains separation representation for this algebra, while Section 6 defines and solves the $B_2^{(1)}$ -type integrable system. In Section 7 the 2×2 Lax matrix $T(u)$ for the $\mathfrak{o}(4)$ Kowalevski top is given which follows by comparing the separation data for two integrable systems from the previous Sections (KT and the integrable system on the quadratic algebra \mathcal{B}). The final result is given in terms of initial variables of the top.

1 Integration in quadratures of the $\mathfrak{o}(4)$ Kowalevski top

In this Section we collect the results from [23, 19] about integration in quadratures of the Kowalevski top on the $\mathfrak{o}(4)$ algebra.

The Poisson brackets for the $\mathfrak{o}(4)$ generators J_k , x_k , $k = 1, 2, 3$, are defined in the standard way:

$$\{J_i, J_k\} = \varepsilon_{ikl} J_l, \quad \{J_i, x_k\} = \varepsilon_{ikl} x_l, \quad \{x_i, x_k\} = -\mathcal{P} \varepsilon_{ikl} J_l, \quad (1)$$

where ε_{ikl} is the completely anti-symmetric tensor, $\varepsilon_{123} = 1$, and \mathcal{P} is a complex (or real) parameter. In fact, one can think of the algebra (1) as a complex $\mathfrak{o}(4, \mathbb{C})$ algebra with the triple of real algebras appearing when the parameter \mathcal{P} is specialized as follows:

$$\mathcal{P} = \begin{cases} -1, & \mathfrak{o}(4) \\ 0, & \mathfrak{e}(3) \\ 1, & \mathfrak{o}(3,1) \end{cases}. \quad (2)$$

The Casimirs of the bracket (1) have the form

$$\ell = x_1 J_1 + x_2 J_2 + x_3 J_3, \quad a^2 = x_1^2 + x_2^2 + x_3^2 - \mathcal{P}(J_1^2 + J_2^2 + J_3^2). \quad (3)$$

The $\mathfrak{o}(4)$ Kowalevski top has the Hamiltonian H ,

$$H = J_1^2 + J_2^2 + 2J_3^2 - 2bx_1, \quad (4)$$

and the second integral K ,

$$K = (J_+^2 + 2bx_+ - \mathcal{P}b^2)(J_-^2 + 2bx_- - \mathcal{P}b^2), \quad (5)$$

$$J_{\pm} = J_1 \pm iJ_2, \quad x_{\pm} = x_1 \pm ix_2, \quad (i^2 = -1),$$

which are Poisson commuting:

$$\{H, K\} = 0. \quad (6)$$

When $\mathcal{P} = 0$, this Liouville integrable system becomes the usual $\mathfrak{e}(3)$ Kowalevski top. The $\mathfrak{o}(4)$ version was proved to be integrable in [14] and it was integrated in [19], where also the separation of variables was performed for this system, with separation equations being put in the form which is best suited for the purpose of this paper.

The integration of the $\mathfrak{o}(4)$ KT in [19] was inspired by the paper [9] about integration of the $\mathfrak{e}(3)$ case where it was done by mapping the KT into the Neumann system, thereby giving another way of looking at the original Kowalevski's transformation. For the $\mathfrak{o}(4)$ case it goes as follows. Introduce new variables:

$$p_1 = \frac{1 + J_+ J_-}{2(J_+ - J_-)}, \quad p_2 = \frac{J_+ + J_-}{2i(J_+ - J_-)}, \quad p_3 = \frac{1 - J_+ J_-}{2i(J_+ - J_-)}, \quad (7)$$

$$l_1 = \frac{J_3(1 - J_+ J_-) - bx_3(J_+ + J_-)}{2(J_+ - J_-)}, \quad l_2 = \frac{ibx_3}{J_+ - J_-}, \quad l_3 = \frac{J_3(1 + J_+ J_-) + bx_3(J_+ + J_-)}{2(J_+ - J_-)}. \quad (8)$$

Under the flow governed by the Kowalevski Hamiltonian,

$$(\dot{\cdot}) \equiv \frac{d(\cdot)}{dt} = \frac{1}{2}\{H, (\cdot)\}, \quad (9)$$

these new variables, p_k, l_k , $k = 1, 2, 3$, combined into two vectors \vec{p} and \vec{l} , evolve as the “ $\mathfrak{e}(3)$ Neumann system”:

$$\dot{\vec{p}} = 2i \vec{l} \times \vec{p}, \quad \dot{\vec{l}} = 2i Q \vec{p} \times \vec{p}, \quad (10)$$

with the symmetric matrix Q depending on Casimirs and both integrals:

$$Q = \begin{pmatrix} -\frac{1}{4} + b^2 \tilde{a} & ib\ell & i(\frac{1}{4} + b^2 \tilde{a}) \\ ib\ell & -\frac{H}{2} & -b\ell \\ i(\frac{1}{4} + b^2 \tilde{a}) & -b\ell & \frac{1}{4} - b^2 \tilde{a} \end{pmatrix}, \quad (11)$$

$$\tilde{a} = a^2 - K/(4b^2) + \mathcal{P}H/2 + b^2\mathcal{P}^2/4. \quad (12)$$

The Neumann system (10)–(12) has four integrals of motion:

$$I_1 = \vec{l}^2 + (Q\vec{p}, \vec{p}), \quad I_2 = (Q\vec{l}, \vec{l}) - (Q^\wedge \vec{p}, \vec{p}), \quad C_1 = (\vec{l}, \vec{p}) = 0, \quad C_2 = \vec{p}^2 = -\frac{1}{4}, \quad (13)$$

where Q^\wedge denotes the adjoint matrix. It is usually formulated together with the canonical $\mathfrak{e}(3)$ Poisson bracket for the generators p_k and l_k . Unfortunately, the transformation (7)–(8) is non-canonical, so it does not bring the $\mathfrak{o}(4)$ bracket of the initial variables into this $\mathfrak{e}(3)$ bracket of the Neumann system. This means we can use the equations (10) only keeping in mind that we have a new Poisson structure on the variables p_k and l_k imposed by the Kowalevski $\mathfrak{o}(4)$ Poisson structure.

As was already mentioned, the mapping of the $\mathfrak{e}(3)$ Kowalevski dynamics ($\mathcal{P} = 0$) into Neumann system's dynamics was first found in [9]. Algebraic geometric arguments proving the existence of such (non-canonical) maps in other ‘genus 2’ situations was also studied in detail around that time (cf. [5] and references therein) producing many other maps between various systems, in particular, tops.

Now, let us proceed further to the integration in quadratures. Since the problem has been reduced to the dynamics of the Neumann system, its integration is well-known (see, for instance, [33]). One introduces two new variables λ_1 and λ_2 as zeros of the polynomial:

$$\lambda^2 + \lambda \left(\frac{H}{2} - 4(Q\vec{p}, \vec{p}) \right) - 4(Q^\wedge \vec{p}, \vec{p}) = (\lambda - \lambda_1)(\lambda - \lambda_2). \quad (14)$$

Then one checks that they satisfy the following equations:

$$\dot{\lambda}_1(\lambda_1 - \lambda_2) = 2\sqrt{-R_5(\lambda_1)}, \quad \dot{\lambda}_2(\lambda_2 - \lambda_1) = 2\sqrt{-R_5(\lambda_2)}, \quad (15)$$

$$R_5(\lambda) = \left((\lambda - \mathcal{P}b^2/2)^2 - K/4 \right) \left[\left(\lambda + \frac{H}{2} \right) (\lambda^2 + b^2\tilde{a}) - b^2\ell^2 \right]. \quad (16)$$

After rewriting these equations in the form

$$\frac{d\lambda_1}{\sqrt{-R_5(\lambda_1)}} + \frac{d\lambda_2}{\sqrt{-R_5(\lambda_2)}} = 0, \quad \frac{\lambda_1 d\lambda_1}{\sqrt{-R_5(\lambda_1)}} + \frac{\lambda_2 d\lambda_2}{\sqrt{-R_5(\lambda_2)}} = 2dt, \quad (17)$$

one uses Abel-Jacobi map to integrate the dynamics in terms of theta-functions. This is exactly what Kowalevski did in 1889 (for $\mathcal{P} = 0$): she found the variables λ_1 and λ_2 (14), she derived the equations (15)–(16), now known as the Kowalevski equations, and she found theta-function formulas for the initial variables of the top, J_k and x_k , as functions of time t . Of course, she was not aware of the ‘simplifying map’ to the Neumann system discovered 100 years later, this is why her original calculations looked so complicated, mysterious and attractive for many generations of mathematicians, and they still do!

Formulas (14)–(17) for arbitrary \mathcal{P} were derived in [19].

2 Separation of variables for the $\mathfrak{o}(4)$ Kowalevski top

In this Section we collect the results from [19] about separation of variables (SoV) for the Kowalevski top on the $\mathfrak{o}(4)$ algebra.

Separation variables have been generally used to construct analytic expressions for the action variables (in terms of abelian integrals) or in order to get a separated representation for the action function. Therefore, the method of separation of variables for a long time served an important, but technical role in solving Liouville integrable systems of classical mechanics. A new, and much more exciting, application of the method came with the development of quantum integrable systems. Because of the fact that quantization of the action variables seems to be a rather formidable task, quantum separation of variables became an inevitable refuge. In fact, it was successfully performed for many families of integrable systems (see, for instance, survey [39]).

We should start from a (working) definition of the SoV. Historically, several very different definitions have been given, each depending on the context. We will always mean the context of finite-dimensional integrable Hamiltonian dynamics or, in other words, our definition will only be valid for the Liouville integrable systems.

By *separation of variables* we mean a canonical transformation to new variables u_j, v_j , $j = 1, \dots, n$, which satisfy the following *separation equations*

$$\sum_{j=1}^n a_{ij} H_j = b_i, \quad i = 1, \dots, n, \quad (18)$$

$$a_{ij} = a_{ij}(u_i, v_i), \quad b_i = b_i(u_i, v_i), \quad (19)$$

or, in other words, in terms of which the integrals of motion H_j acquire the following ‘separated form’:

$$\vec{H} = A^{-1} \vec{B}, \quad (A)_{ij} = a_{ij}, \quad (\vec{B})_i = b_i, \quad (\vec{H})_i = H_i. \quad (20)$$

The conditions (19) that the functions a_{ij} and b_i in (18) depend on the new (separation) variables with the index i *only*, is crucial. It indeed means that the n equations in (18) are really separated from one another.

The above definition includes, as a special canonical transform, the (classical) coordinate separation of variables, when the new coordinates, say u_j , are the functions of the old coordinates (q_j) only, and they do not depend on the momenta (p_j). See the book [13] about the history of this sub-class of transformations with many examples of such situation.

General (separating) canonical transforms, however, result in new (separation) variables being non-trivial functions of all $2n$ initial canonical variables,

$$u_j = u_j(q_1, \dots, q_n, p_1, \dots, p_n), \quad v_j = v_j(q_1, \dots, q_n, p_1, \dots, p_n), \quad j = 1, \dots, n, \quad (21)$$

$$\{u_j, v_k\} = 0, \quad j \neq k, \quad \{v_j, u_j\} = 1, \quad j = 1, \dots, n. \quad (22)$$

The examples of such separating canonical transforms are usually much more sophisticated than those of the coordinate ones. As far as I know, the first explicit example was given by van Moerbeke in [32] concerning separation of variables for the Toda lattice (see also [8]). The method was further developed by Komarov in a series of works on tops, including quantum separation of variables, see [15, 16]. Many further examples have been produced since 1982, with the theory benefiting mostly from the developments of the algebraic geometric and r -matrix understanding of the method of separation of variables. This led to a rather satisfying picture of the present state-of-art of non-coordinate separation of variables. See [39] and [11, 12] for more details.

It is interesting to remark that many explicit non-coordinate separations were already produced by the classics, such as those for the Goryachev-Chaplygin and Kowalevski top,

but they never calculated the Poisson brackets, so that many modern developments involved checking the canonicity of the transformations, proposed by mathematicians of the 19th century! Hence, classics *knew* about non-coordinate separation of variables (sometimes more than we do now). It is a pity that we lack a good review of this subject which would cover, say, the separation of variables for the Clebsch, Euler-Manakov, Steklov and many other tops, including modern results and those obtained by classics.

Two pairs of separation variables for the $\mathfrak{o}(4)$ Kowalevski top are as follows:

$$s_i = 2\lambda_i + H, \quad p_i = \frac{1}{2\sqrt{-2s_i}} \ln \frac{\xi_i + \sqrt{\xi_i^2 + d_i^2}}{d_i}, \quad i = 1, 2, \quad (23)$$

$$\xi_i = 2\sqrt{y_i^2 + d_i y_i}, \quad y_i = (s_i - H - \mathcal{P}b^2)^2 - K, \quad d_i = 4b^2 \left(\frac{\mathcal{P}s_i}{2} + a^2 - \frac{2\ell^2}{s_i} \right). \quad (24)$$

They are canonical variables,

$$\{s_1, s_2\} = \{p_1, p_2\} = \{s_1, p_2\} = \{s_2, p_1\} = 0, \quad (25)$$

$$\{p_1, s_1\} = 1, \quad \{p_2, s_2\} = 1, \quad (26)$$

and they satisfy the following separation equations:

$$s_i^3 - (2H + \mathcal{P}b^2)s_i^2 + \kappa s_i - 4b^2\ell^2 = 2b^2 \left(\frac{\mathcal{P}s_i^2}{2} + a^2 s_i - 2\ell^2 \right) \cos(2\sqrt{2s_i}p_i), \quad (27)$$

$$\kappa = (H + \mathcal{P}b^2)^2 - K + 2a^2b^2. \quad (28)$$

Notice that the role of the H_1 and H_2 from the definition (18) is played here by the integrals H and κ . The functions a_{ij} and b_i can be directly read from the formulas (27). Also notice that the formulas (27) are equivalent to two formulas for p_i 's from (23).

This result, i.e. the separation of variables, was given for $\mathcal{P} = 0$ in [41]¹ and for the general case in [19]. We will give here a shorter proof than the one in [19], but before doing that let us reformulate the statement in terms of other variables, the s_i and y_i .

The variables s_i and y_i , $i = 1, 2$, are ‘almost canonical’,

$$\{s_1, s_2\} = \{y_1, y_2\} = \{s_1, y_2\} = \{s_2, y_1\} = 0, \quad (29)$$

$$\{y_1, s_1\} = -4\sqrt{-2s_1y_1(y_1 + d_1)}, \quad \{y_2, s_2\} = -4\sqrt{-2s_2y_2(y_2 + d_2)}, \quad (30)$$

and they satisfy the following simple separation equations:

$$y_i = (s_i - H - \mathcal{P}b^2)^2 - K, \quad i = 1, 2. \quad (31)$$

In order to bring the equations (29)–(31) into the equations (25)–(28) one needs to make a transformation from y_i 's to the canonically conjugated variables p_i 's which is easily found by taking the integral,

$$p_i = -\frac{1}{4} \int^{y_i} \frac{dy}{\sqrt{-2s_i y(y + d_i)}} = \frac{1}{2\sqrt{-2s_i}} \ln \left(1 + \frac{2y_i}{d_i} + 2\sqrt{\frac{y_i}{d_i} \left(\frac{y_i}{d_i} + 1 \right)} \right). \quad (32)$$

¹without a proof

The separation equations (31) are then transformed into the equations

$$s_i \left(y_i + \frac{d_i}{2} \right) = \frac{s_i d_i}{2} \cos(2\sqrt{2s_i} p_i), \quad (33)$$

which are equivalent to the separation equations (27). Notice here that $R_5(\lambda_i)$ from (15)–(16) have the following expressions in terms of the variables s_i , y_i and d_i :

$$R_5(\lambda_i) = \frac{s_i y_i}{32} (y_i + d_i), \quad i = 1, 2. \quad (34)$$

Finally, the brackets (29)–(30) are checked by a direct computation².

3 Hyperelliptic Prymian and ansatz for the Lax matrix

In this Section we change the separation variables s_i and p_i to new separation variables u_i and m_i^\pm , which respect the symmetry of the problem, and while doing that we will derive the proper ansatz for the future 2×2 Lax matrix of the $\mathfrak{o}(4)$ Kowalevski top.

First of all, notice that in the definition of p_i 's in (23) there appears $\sqrt{s_i}$, which is very strange taking into account that $\lambda = -\frac{H}{2}$ (which is equivalent to $s = 0$ according to the definition of s -variables (23)) is *not* a branching point of the Kowalevski curve $\mu^2 = R_5(\lambda)$ (cf. (16)).

This ‘paradox’ was explained in [19] by stating that the s -variables are *not* the proper variables and one should introduce new, u -variables, which respect the additional symmetry present in the problem³. This problem of s -variables passed unnoticed in [41] in the $\mathfrak{e}(3)$ case.

The proper new variables u_i and \hat{y}_i , $i = 1, 2$, are introduced as follows:

$$u_i := \sqrt{\frac{s_i}{2}}, \quad \hat{y}_i := (u_i^2 - H/2 - \mathcal{P}b^2/2)^2 - K/4, \quad i = 1, 2. \quad (35)$$

Another pair of variables, which are functions of u -variables and Casimirs, will be used:

$$\hat{d}_i := b^2 \left(\mathcal{P}u_i^2 + a^2 - \frac{\ell^2}{u_i^2} \right), \quad i = 1, 2. \quad (36)$$

The variables u_i and \hat{y}_i are ‘almost canonical’:

$$\{u_1, u_2\} = \{\hat{y}_1, \hat{y}_2\} = \{u_1, \hat{y}_2\} = \{u_2, \hat{y}_1\} = 0, \quad (37)$$

$$\{\hat{y}_1, u_1\} = -2\sqrt{-\hat{y}_1(\hat{y}_1 + \hat{d}_1)}, \quad \{\hat{y}_2, u_2\} = -2\sqrt{-\hat{y}_2(\hat{y}_2 + \hat{d}_2)}. \quad (38)$$

Defining new variables m_i^\pm ,

$$m_i^\pm := 1 + 2 \frac{\hat{y}_i}{\hat{d}_i} \pm 2 \sqrt{\frac{\hat{y}_i}{\hat{d}_i} \left(\frac{\hat{y}_i}{\hat{d}_i} + 1 \right)}, \quad i = 1, 2, \quad (39)$$

²on a computer: checking (29) takes about 2 hours

³ for the usual $\mathfrak{e}(3)$ Kowalevski top, i.e. in the $\mathcal{P} = 0$ case, the original explanation was presented much earlier in author’s MSc dissertation in 1985, see [22], [18]

we finally obtain the separation equations in the form

$$u_i^6 - (H + \mathcal{P}b^2/2)u_i^4 + \frac{1}{4}((H + \mathcal{P}b^2)^2 - K + 2a^2b^2)u_i^2 - \frac{b^2\ell^2}{2} = \frac{b^2}{4}(\mathcal{P}u_i^4 + a^2u_i^2 - \ell^2)(m_i^+ + m_i^-). \quad (40)$$

The final set of separation variables u_i and m_i^\pm satisfy two separation equations above and the algebra below:

$$\{u_1, u_2\} = 0, \quad \{u_j, m_k^\pm\} = 0, \quad j \neq k, \quad (41)$$

$$\{m_j^\pm, u_j\} = \mp 2i m_j^\pm, \quad m_j^+ m_j^- = 1. \quad (42)$$

As was already mentioned, this was a result of [18] and [19] in the $e(3)$ and $o(4)$ case, respectively. In [22, 18] these separation variables and corresponding action variables were used for constructing the quasiclassical spectrum of the integrals of motion of the Kowalevski top.

If we exclude the variables m_i^+ using the condition that $m_i^+ m_i^- = 1$ and if we rescale the variables m_i^- ,

$$m_i^+ = \frac{1}{m_i^-}, \quad \widetilde{m}_i^- = \frac{b^2}{4}(\mathcal{P}u_i^4 + a^2u_i^2 - \ell^2) m_i^-, \quad (43)$$

the separation equations (40) will acquire the following form:

$$\left(\widetilde{m}_i^-\right)^2 - P_3(u_i^2) \widetilde{m}_i^- + \left(P_2(u_i^2)\right)^2 = 0, \quad (44)$$

where two polynomials $P_3(u)$ and $P_2(u)$ are

$$P_3(u) = u^3 - (H + \mathcal{P}b^2/2)u^2 + \frac{1}{4}((H + \mathcal{P}b^2)^2 - K + 2a^2b^2)u - \frac{b^2\ell^2}{2}, \quad (45)$$

$$P_2(u) = \frac{b^2}{4}(\mathcal{P}u^2 + a^2u - \ell^2). \quad (46)$$

Therefore, separation equations (40) amount to having two points, (u_1, \widetilde{m}_1^-) and (u_2, \widetilde{m}_2^-) , on the algebraic curve Γ :

$$\Gamma : \quad m^2 - P_3(u^2)m + \left(P_2(u^2)\right)^2 = 0, \quad (47)$$

$$(u_i, \widetilde{m}_i^-) \in \Gamma, \quad i = 1, 2. \quad (48)$$

The curve Γ (47) is a hyperelliptic curve with the involution $u \mapsto -u$. This is the curve that replaces Kowalevski's genus 2 hyperelliptic curve $\mu^2 = R_5(\lambda)$ if one takes into account the Poisson structure and the symmetry of the problem. The Kowalevski dynamics is therefore linearized on the corresponding hyperelliptic Prymian of the curve Γ^4 . We can bring the curve Γ (47) into the standard hyperelliptic form by shifting the variable m :

$$m \mapsto \hat{m} = m - \frac{1}{2} P_3(u^2), \quad (49)$$

⁴ see the survey by Markushevich in this volume (and also his work [31]) about interrelations between different curves for the Kowalevski top in the case of $e(3)$ algebra and zero Casimir $\ell = 0$

thus getting the curve,

$$\hat{m}^2 = \frac{1}{4} \left(P_3(u^2) - 2P_2(u^2) \right) \left(P_3(u^2) + 2P_2(u^2) \right) \equiv \frac{1}{4} R_6(u^2), \quad (50)$$

$$R_6(u) = u R_5(u - H/2), \quad (51)$$

with Kowalevski's polynomial $R_5(\lambda)$ given in (16).

Now we can formulate the problem: *to find a 2×2 Lax matrix for which the curve Γ (47) is the spectral curve or, in other words, to find the proper 2×2 Lax matrix for the Kowalevski top which is related to Kowalevski's separation of variables.*

We must mention here the known Lax matrices for the Kowalevski top. Let us consider only the e(3) case, as nothing is known about Lax matrices for the o(4) case. Previous attempts to construct Lax matrices for the KT included: (i) 2×2 matrix in [7], (ii) 3×3 matrix in [9], (iii) 4×4 and 6×6 matrices in [5], and (iv) 5×5 (or 4×4) matrix in [35]. The first three do not respect the Poisson structure of the problem in contrast to the last one, which satisfies a linear r -matrix algebra. Unfortunately, no separation of variables is known which can be related to the Lax matrix of [35]. In fact, as I said before, no separation is known which is alternative to the original Kowalevski's separation of the problem!

What can be guessed about the ansatz for the required 2×2 Lax matrix $T(u)$? From the form of the spectral curve Γ (47) one can conclude that the entries of $T(u)$ will be polynomials in u of order not higher than 6, the $\text{tr } T(u)$ will be equal to $P_3(u^2)$ and the $\det T(u)$ will be equal to $(P_2(u^2))^2$ and, hence, will depend only on Casimirs. In the next Sections we will construct (and solve) an integrable system whose Lax matrix has exactly these properties. This system will be formulated within the framework of the quadratic r -matrix algebra for integrable systems with boundary conditions. The o(4) Kowalevski top, as we will see, will correspond to a system with $B_2^{(1)}$ -type boundary conditions.

4 Quadratic algebra \mathcal{B}

Starting from about 1982 the method of separation of variables gets connected with the r -matrix formalism of the quantum inverse scattering method, developed during that time by the Leningrad School. It was noticed by Komarov (see the footnote in [36] and a full credit in [39]) that for the 2×2 L -operators (Lax matrices) the separation variables ought to be the zeros of the off-diagonal element of the L -operator. This observation was fully exploited by Sklyanin in [36, 37] who developed a beautiful (pure algebraic) setting for the method within the framework of the r -matrix technique. Since then this approach took off and led to separations for many families of integrable systems. Sklyanin also generalized the approach to include higher rank L -operators and non-standard normalizations. See the review [39] where many of the examples were exposed. An alternative, algebraic geometric approach, which dates back to Adler and van Moerbeke [3, 4] and Mumford [33] and includes many researchers, have been developed starting from about the same time (see, for instance, [34, 10, 1, 2, 40, 11, 12]). It also led to many important new separations for complicated integrable systems and tops. Unfortunately, we can not review this another approach, as it would require much larger scope than the pure algebraic one we adopted here.

In [36] the Goryachev-Chaplygin top and in [28] the (symmetric) Neumann system and so-called Kowalevski-Chaplygin-Goryachev top were related to special representations of the

quadratic r -matrix algebras. In the present paper we show how the $\mathfrak{o}(4)$ Kowalevski top is related to a special representation of the quadratic r -matrix algebra for integrable systems with boundary conditions. We start by introducing the corresponding quadratic algebra \mathcal{B} .

Let $\delta \in \mathbb{C}$. Consider the following L -operator:

$$L(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}, \quad (52)$$

$$A(u) = A_4 u^4 + A_3 u^3 + A_2 u^2 + A_1 u + \delta, \quad (53)$$

$$D(u) = A(-u), \quad (54)$$

$$B(u) = B_3 u^3 + B_1 u, \quad (55)$$

$$C(u) = u^5 + C_3 u^3 + C_1 u, \quad (56)$$

$$\deg_u L(u) = \begin{pmatrix} 4 & 3 \\ 5 & 4 \end{pmatrix}, \quad L(-u) = L^\wedge(u). \quad (57)$$

Introduce B_2 -type quadratic Poisson algebra \mathcal{B} with eight generators, $A_1, \dots, A_4, B_1, B_3, C_1, C_3$, and the following Poisson brackets:

$$\{A_4, A_3\} = 2i B_3, \quad \{A_4, A_2\} = 0, \quad \{A_4, A_1\} = 2i B_1, \quad (58)$$

$$\{A_3, A_2\} = -2i B_1, \quad \{A_3, A_1\} = 0, \quad \{A_2, A_1\} = 2i (B_1 C_3 - B_3 C_1), \quad (59)$$

$$\{B_3, B_1\} = 0, \quad \{C_3, C_1\} = 0, \quad (60)$$

$$\{B_3, C_3\} = -8i A_3 A_4, \quad \{B_3, C_1\} = -8i A_1 A_4, \quad (61)$$

$$\{B_1, C_3\} = -8i A_1 A_4, \quad \{B_1, C_1\} = -8i (-\delta A_3 + A_1 A_2), \quad (62)$$

$$\{B_3, A_4\} = 0, \quad \{B_3, A_3\} = 4i B_3 A_4, \quad (63)$$

$$\{B_3, A_2\} = 0, \quad \{B_3, A_1\} = 4i B_1 A_4, \quad (64)$$

$$\{B_1, A_4\} = 0, \quad \{B_1, A_3\} = 4i B_1 A_4, \quad (65)$$

$$\{B_1, A_2\} = 4i (B_1 A_3 - B_3 A_1), \quad \{B_1, A_1\} = 4i (B_1 A_2 - \delta B_3), \quad (66)$$

$$\{C_3, A_4\} = 4i A_3, \quad \{C_3, A_3\} = 4i (A_2 - C_3 A_4), \quad (67)$$

$$\{C_3, A_2\} = 4i A_1, \quad \{C_3, A_1\} = 4i (\delta - C_1 A_4), \quad (68)$$

$$\{C_1, A_4\} = 4i A_1, \quad \{C_1, A_3\} = 4i (\delta - C_1 A_4), \quad (69)$$

$$\{C_1, A_2\} = 4i (A_1 C_3 - A_3 C_1), \quad \{C_1, A_1\} = 4i (\delta C_3 - C_1 A_2). \quad (70)$$

This is a rank 2 algebra because there are four Casimirs Q_1, \dots, Q_4 , which are the coefficients of the $\det L(u)$:

$$\det L(u) = Q_4 u^8 + Q_3 u^6 + Q_2 u^4 + Q_1 u^2 + \delta^2, \quad (71)$$

$$Q_4 = A_4^2 - B_3, \quad (72)$$

$$Q_3 = 2A_2 A_4 - A_3^2 - B_3 C_3 - B_1, \quad (73)$$

$$Q_2 = A_2^2 + 2\delta A_4 - 2A_1 A_3 - B_3 C_1 - B_1 C_3, \quad (74)$$

$$Q_1 = 2\delta A_2 - A_1^2 - B_1 C_1. \quad (75)$$

In the matrix notations, this algebra looks very compact,

$$\{L_1(u), L_2(v)\} = [r(u-v), L_1(u)L_2(v)] + L_1(u)r(u+v)L_2(v) - L_2(v)r(u+v)L_1(u), \quad (76)$$

with the r -matrix $r(u)$ being as follows:

$$r(u) = \frac{-2i}{u} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (77)$$

In (76) we use the standard notations for the tensor products,

$$L_1(u) = L(u) \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad L_2(v) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes L(v). \quad (78)$$

Also, in the left-hand-side of (76) we have a 4×4 matrix with the entries $(\{L_1(u), L_2(v)\})_{ij,kl} \equiv \{(L(u))_{ij}, (L(v))_{kl}\}$ and in the right-hand-side of (76) there are (4×4) matrix commutator and products. For more details on these notations and on quadratic r -matrix algebras see [36, 37, 38, 28, 29, 24, 30, 20, 27, 39, 25, 26].

The Poisson brackets for the polynomials $A(u)$, $B(u)$ and $C(u)$ (the polynomial $D(u) \equiv A(-u)$) read

$$\{A(u), A(v)\} = \frac{-2i}{u+v} (B(u)C(v) - B(v)C(u)), \quad \{B(u), B(v)\} = 0, \quad \{C(u), C(v)\} = 0, \quad (79)$$

$$\{B(u), A(v)\} = \frac{-2i}{u-v} (B(u)A(v) - B(v)A(u)) + \frac{2i}{u+v} (A(v)B(u) + B(v)A(-u)), \quad (80)$$

$$\{C(u), A(v)\} = \frac{-2i}{u-v} (A(u)C(v) - A(v)C(u)) - \frac{2i}{u+v} (C(u)A(v) + A(-u)C(v)), \quad (81)$$

$$\{B(u), C(v)\} = \frac{-2i}{u-v} (A(-u)A(v) - A(-v)A(u)) - \frac{2i}{u+v} (A(u)A(v) - A(-v)A(-u)). \quad (82)$$

5 Separation representation of the algebra \mathcal{B}

Let us realize the algebra \mathcal{B} (58)–(70) in terms of ‘separation variables’ \hat{u}_j , \hat{m}_j^\pm , $j = 1, 2$.

First, introduce two new Poisson commuting variables \hat{u}_1 and \hat{u}_2 as zeros of the polynomial $C(u)$:

$$C(u) = u^5 + C_3 u^3 + C_1 u = u(u^2 - \hat{u}_1^2)(u^2 - \hat{u}_2^2), \quad (83)$$

$$\hat{u}_1^2 + \hat{u}_2^2 = -C_3, \quad \hat{u}_1^2 \hat{u}_2^2 = C_1, \quad (84)$$

$$\{C_3, C_1\} = 0 \quad \Leftrightarrow \quad \{\hat{u}_1, \hat{u}_2\} = 0. \quad (85)$$

The ‘conjugated’ variables \hat{m}_j^\pm are introduced as corresponding values of the polynomial $A(\mp u)$ at these \hat{u}_j :

$$\hat{m}_j^\pm = A(\mp \hat{u}_j), \quad j = 1, 2. \quad (86)$$

It is easy to prove that these new variables obey the following (separated) algebra of Poisson brackets:

$$(A) \quad \hat{m}_j^+ \hat{m}_j^- = \det L(\hat{u}_j), \quad (87)$$

$$(B) \quad \{\hat{m}_j^\pm, \hat{u}_j\} = \mp 2i \hat{m}_j^\pm, \quad j = 1, 2, \quad (88)$$

$$(C) \quad \{\hat{m}_k^\pm, \hat{u}_j\} = \{\hat{m}_k^\pm, \hat{m}_j^\pm\} = \{\hat{u}_k, \hat{u}_j\} = 0, \quad k \neq j. \quad (89)$$

Let us prove, for instance, that $\{\hat{u}_j, \hat{m}_k^-\} = -2i \delta_{jk} \hat{m}_k^-$ (cf. [36, 24]). Notice first that by the differentiation property of the Poisson bracket,

$$0 \equiv \{C(\hat{u}_j), A(v)\} = C'(\hat{u}_j) \{\hat{u}_j, A(v)\} + \{C(u), A(v)\}|_{u=\hat{u}_j}. \quad (90)$$

Hence,

$$\{\hat{u}_j, A(v)\} = -\frac{1}{C'(\hat{u}_j)} \{C(u), A(v)\}|_{u=\hat{u}_j}. \quad (91)$$

Now, substituting $u = \hat{u}_j$ into the algebraic relation (81), we obtain

$$\{C(u), A(v)\}|_{u=\hat{u}_j} = \frac{-2i}{\hat{u}_j - v} \hat{m}_j^- C(v) - \frac{2i}{\hat{u}_j + v} \hat{m}_j^+ C(v). \quad (92)$$

Therefore,

$$\{\hat{u}_j, \hat{m}_k^-\} = 0, \quad j \neq k, \quad (93)$$

and

$$\{\hat{u}_j, \hat{m}_j^-\} = -2i \hat{m}_j^-, \quad j = 1, 2. \quad (94)$$

The rest of the relations (A), (B), (C) can be proved in the similar way.

Now we can realize the four polynomials $A(u)$, $B(u)$, $C(u)$ and $D(u)$, whose coefficients are the generators of the algebra \mathcal{B} , in terms of \hat{u}_j , \hat{m}_j^\pm , $j = 1, 2$, from the following data:

- $C_3 = -\hat{u}_1^2 - \hat{u}_2^2, \quad C_1 = \hat{u}_1^2 \hat{u}_2^2 \Rightarrow C(u)$
- $A(\pm \hat{u}_j) = \hat{m}_j^\mp, \quad A(0) = \delta \Rightarrow A(u)$
- $D(u) = A(-u)$
- $B(u) = \frac{A(u)D(u) - \det L(u)}{C(u)}$, $C(u)$ and the numerator have common zeros at $u = 0$ and at $u = \pm \hat{u}_{1,2}$, so that $C(u)$ divides the numerator.

The result is given by the formulas

$$\begin{aligned} A(u) = & \frac{(u^2 - \hat{u}_1^2)(u^2 - \hat{u}_2^2)}{\hat{u}_1^2 \hat{u}_2^2} \delta + \frac{u(u^2 - \hat{u}_1^2) \left((u - \hat{u}_2) \hat{m}_2^+ + (u + \hat{u}_2) \hat{m}_2^- \right)}{2\hat{u}_2^2(\hat{u}_2^2 - \hat{u}_1^2)} \\ & + \frac{u(u^2 - \hat{u}_2^2) \left((u - \hat{u}_1) \hat{m}_1^+ + (u + \hat{u}_1) \hat{m}_1^- \right)}{2\hat{u}_1^2(\hat{u}_1^2 - \hat{u}_2^2)}, \quad D(u) = A(-u), \end{aligned} \quad (95)$$

$$C(u) = u(u^2 - \hat{u}_1^2)(u^2 - \hat{u}_2^2), \quad B(u) = B_3 u^3 + B_1 u, \quad (96)$$

$$B_3 = A_4^2 - q_4, \quad B_1 = 2A_2 A_4 - A_3^2 + (\hat{u}_1^2 + \hat{u}_2^2)(A_4^2 - q_4) - q_3, \quad (97)$$

where q_4 and q_3 are fixed values of the Casimirs from the determinant (cf. (71)),

$$\det L(u) = q_4 u^8 + q_3 u^6 + q_2 u^4 + q_1 u^2 + \delta^2. \quad (98)$$

6 $B_2^{(1)}$ -type integrable system

Let us now introduce an integrable system (of $B_2^{(1)}$ -type) on the algebra \mathcal{B} . Denote by $K(u)$ the following constant representation of the algebra (76) with the r -matrix (77):

$$K(u) = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}. \quad (99)$$

Define the integrable system by its Lax matrix $T(u)$,

$$T(u) := K(u)L(u). \quad (100)$$

Its integrals of motion are the coefficients of the trace of the Lax matrix $T(u)$ ($\det T(u)$ is a Casimir),

$$\mathrm{tr} T(u) = u^6 + \mathcal{H}_1 u^4 + \mathcal{H}_2 u^2 + 2\delta, \quad (101)$$

$$\mathcal{H}_1 = 2A_4 + C_3, \quad \mathcal{H}_2 = 2A_2 + C_1, \quad \{\mathcal{H}_1, \mathcal{H}_2\} = 0. \quad (102)$$

The variables $\hat{u}_j, \hat{m}_j^\pm, j = 1, 2$, from the previous Section *are* the separation variables for the integrable system (102). Indeed, because $\mathrm{tr} T(u) = A(u) + D(u) + uC(u)$, one obtains the following separation equations:

$$\mathrm{tr} T(\hat{u}_j) = \hat{u}_j^6 + \mathcal{H}_1 \hat{u}_j^4 + \mathcal{H}_2 \hat{u}_j^2 + 2\delta = \hat{m}_j^+ + \hat{m}_j^-, \quad j = 1, 2. \quad (103)$$

This is the separation of variables for the $B_2^{(1)}$ -type integrable system on the algebra \mathcal{B} with the integrals of motion \mathcal{H}_1 and \mathcal{H}_2 (102).

Notice here that a general constant representation of the algebra (76) with the r -matrix (77) is given by the full matrix,

$$\hat{K}(u) = \begin{pmatrix} \alpha u + \delta & \beta u \\ \gamma u & -\alpha u + \delta \end{pmatrix}, \quad (104)$$

which is used to generate integrable systems with more general boundary conditions, of BC -type (see, for instance, the examples of Toda lattice in [38] and Kowalevski-Chaplygin-Goryachev top in [28]). The D -type boundary conditions for the Toda lattice are described by a non-constant matrix $\hat{K}(u)$ depending on the dynamical variables (cf. [29, 24, 30, 20, 27, 25, 26]).

7 Lax matrix for the $\mathfrak{o}(4)$ Kowalevski top

Now we have everything to write down the 2×2 Lax matrix $T(u)$ for the $\mathfrak{o}(4)$ KT. It will have the factorization (100),

$$T(u) = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}, \quad (105)$$

and we only have to find the expressions for the polynomials $A(u), B(u), C(u)$ and $D(u)$ in terms of the $\mathfrak{o}(4)$ variables J_k and $x_k, k = 1, 2, 3$.

Notice that two considered above integrable systems, the $\mathfrak{o}(4)$ KT and the $B_2^{(1)}$ -type integrable system on the quadratic algebra \mathcal{B} , look the same in terms of the separation

variables, so that one can identify their separation variables and separation equations as follows:

$$\hat{u}_j = u_j, \quad \hat{m}_j^\pm = P_2(u_j^2) m_j^\pm, \quad j = 1, 2, \quad (106)$$

$$P_3(u^2) = \text{tr } T(u) = u^6 + \mathcal{H}_1 u^4 + \mathcal{H}_2 u^2 + 2\delta, \quad (107)$$

$$\left(P_2(u^2)\right)^2 = \det T(u) = q_4 u^8 + q_3 u^6 + q_2 u^4 + q_1 u^2 + \delta^2, \quad (108)$$

where polynomials $P_3(u)$ and $P_2(u)$ are given in (45)–(46). To see that, compare the separation equations (103) and the ones for the $\mathfrak{o}(4)$ KT (40).

As a direct consequence, we can restore the whole algebra, i.e. get a new representation of the algebra \mathcal{B} in terms of the $\mathfrak{o}(4)$ variables. The result is as follows:

$$A_4 = \frac{1}{2} \left(u_1^2 + u_2^2 - H - \mathcal{P}b^2/2 \right) = \frac{\hat{y}_1 - \hat{y}_2}{2(u_1^2 - u_2^2)} + \frac{\mathcal{P}b^2}{4}, \quad (109)$$

$$A_2 = \frac{1}{8} \left((H + \mathcal{P}b^2)^2 - K + 2a^2b^2 - 4u_1^2u_2^2 \right) = \frac{\hat{y}_2u_1^2 - \hat{y}_1u_2^2}{2(u_1^2 - u_2^2)} + \frac{a^2b^2}{4}, \quad (110)$$

$$A_0 = -\frac{b^2\ell^2}{4}, \quad A_3 = \frac{i}{8} \{H, u_1^2 + u_2^2\}, \quad A_1 = -\frac{i}{8} \{H, u_1^2u_2^2\}, \quad (111)$$

$$C_3 = -u_1^2 - u_2^2, \quad C_1 = u_1^2u_2^2, \quad (112)$$

$$B_3 = \frac{1}{4} (u_1^2 + u_2^2 - H)(u_1^2 + u_2^2 - H - \mathcal{P}b^2) = \frac{\hat{y}_1 - \hat{y}_2}{4(u_1^2 - u_2^2)} \left(\frac{\hat{y}_1 - \hat{y}_2}{u_1^2 - u_2^2} + \mathcal{P}b^2 \right), \quad (113)$$

$$B_1 = 2A_2A_4 - A_3^2 + (u_1^2 + u_2^2)B_3 - \mathcal{P}a^2b^4/8. \quad (114)$$

Or, explicitly,

$$A_4 = -\frac{1}{2} \left(X^2 + J_3^2 - \mathcal{P}b^2/2 \right), \quad (115)$$

$$A_2 = \frac{J^2}{2} \left(X^2 + J_3^2 \right) + bx_2J_3X - bx_1J_3^2 + \frac{b^2}{2} \left(\mathcal{P}J_3^2 - x_3^2 + \frac{a^2}{2} \right), \quad (116)$$

$$A_0 = -\frac{b^2\ell^2}{4}, \quad A_3 = -\frac{i}{2} \left(X^3 + (J_3^2 + bx_1 - \mathcal{P}b^2)X + bx_2J_3 \right), \quad (117)$$

$$A_1 = \frac{i}{2} \left(J^2X^3 + 2bx_2J_3X^2 + \left(J_3^2(J^2 - bx_1) + b^2(x_2^2 - x_3^2 - \mathcal{P}J_1^2) \right. \right. \\ \left. \left. + bJ_1(x_1J_1 + x_2J_2) \right) X + bx_2J_3(J_2^2 + J_3^2 - bx_1) + J_1J_2J_3(bx_1 - \mathcal{P}b^2) \right), \quad (118)$$

$$C_3 = X^2 - J^2 + 2bx_1 - \mathcal{P}b^2, \quad (119)$$

$$C_1 = -J^2X^2 - 2b(x_2J_3 - x_3J_2)X - 2b\ell J_1 - b^2(x_2^2 + x_3^2 - \mathcal{P}J_1^2), \quad (120)$$

$$B_3 = \frac{1}{4} \left(X^2 + J_3^2 \right) \left(X^2 + J_3^2 - \mathcal{P}b^2 \right), \quad (121)$$

$$B_1 = -\frac{1}{4} \left(J^2X^4 + 2bx_2J_3X^3 + \left(2J_3^2(J^2 - bx_1 + \mathcal{P}b^2/2) + a^2b^2 - b^2(2x_3^2 + x_1^2) \right) X^2 \right. \\ \left. + 2bx_2J_3(J_3^2 - bx_1)X + J_3^4(J^2 - 2bx_1) - b^2J_3^2(x_3^2 - x_1^2 + \mathcal{P}(J_1^2 + J_2^2)) + \mathcal{P}b^4x_3^2 \right), \quad (122)$$

where

$$X := \frac{J_1J_3 + bx_3}{J_2}, \quad J^2 := J_1^2 + J_2^2 + J_3^2. \quad (123)$$

Recall that

$$\ell = x_1 J_1 + x_2 J_2 + x_3 J_3, \quad a^2 = x_1^2 + x_2^2 + x_3^2 - \mathcal{P}J^2. \quad (124)$$

The formulas (115)–(124) give the polynomials $A(u)$, $B(u)$, $C(u)$ and $D(u)$ (cf. (53)–(56)), which together with the formula (105) define the 2×2 Lax matrix $T(u)$ for the $\mathfrak{o}(4)$ Kowalevski top. This Lax matrix satisfies the quadratic (r -matrix) algebra, corresponding to the B_2 -type boundary conditions, and it is the one which is responsible for the separation of variables for the Kowalevski top.

Recall that the spectral curve Γ of the Lax matrix $T(u)$ (105) has the following form:

$$\Gamma : \quad \det(T(u) - m) \equiv m^2 - \operatorname{tr} T(u) m + \det T(u) = 0, \quad (125)$$

$$\operatorname{tr} T(u) = P_3(u^2) = u^6 - (H + \mathcal{P}b^2/2)u^4 + \frac{1}{4}((H + \mathcal{P}b^2)^2 - K + 2a^2b^2)u^2 - \frac{b^2\ell^2}{2}, \quad (126)$$

$$\det T(u) = (P_2(u^2))^2 = \frac{b^4}{16}(\mathcal{P}u^4 + a^2u^2 - \ell^2)^2. \quad (127)$$

Knowing the r -matrix algebraic structure (76)–(77) of the found Lax matrix, it is not difficult to derive the full Lax pair, namely: the Hamiltonian flow given by the Kowalevski Hamiltonian H (cf. (9)),

$$\dot{J}_1 = J_2 J_3, \quad \dot{J}_2 = -J_3 J_1 - bx_3, \quad \dot{J}_3 = bx_2, \quad (128)$$

$$\dot{x}_1 = 2x_2 J_3 - x_3 J_2, \quad \dot{x}_2 = x_3 J_1 - 2x_1 J_3 + b\mathcal{P}J_3, \quad \dot{x}_3 = x_1 J_2 - x_2 J_1 - b\mathcal{P}J_2, \quad (129)$$

has the following Lax pair, $T(u)$ and $M(u)$,

$$\dot{T}(u) = -i[T(u), M(u)], \quad M(u) = \begin{pmatrix} u & 2A_4 \\ 2 & -u \end{pmatrix}. \quad (130)$$

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References

- [1] M. R. Adams, J. Harnad, and J. Hurtubise (1991). *Coadjoint orbits, spectral curves and Darboux coordinates*. In *The geometry of Hamiltonian systems (Berkeley, CA, 1989)*, pp. 9–21. New York: Springer.
- [2] M. R. Adams, J. Harnad, and J. Hurtubise (1993). *Darboux coordinates and Liouville-Arnol'd integration in loop algebras*. Comm. Math. Phys. **155**(2), pp. 385–413.
- [3] M. Adler and P. van Moerbeke (1980a). *Completely integrable systems, Euclidean Lie algebras, and curves*. Adv. in Math. **38**(3), pp. 267–317.
- [4] M. Adler and P. van Moerbeke (1980b). *Linearization of Hamiltonian systems, Jacobi varieties and representation theory*. Adv. in Math. **38**(3), pp. 318–379.

- [5] M. Adler and P. van Moerbeke (1988). *The Kowalewski and Hénon-Heiles motions as Manakov geodesic flows on $SO(4)$ —a two-dimensional family of Lax pairs*. Comm. Math. Phys. **113**(4), pp. 659–700.
- [6] A. I. Bobenko, A. G. Reyman, and M. A. Semenov-Tian-Shansky (1989). *The Kowalewski top 99 years later: a Lax pair, generalizations and explicit solutions*. Comm. Math. Phys. **122**(2), pp. 321–354.
- [7] L. D. Fairbanks (1988). *Lax equation representation of certain completely integrable systems*. Compositio Math. **68**(1), pp. 31–40.
- [8] H. Flaschka and D. W. McLaughlin (1976). *Canonically conjugate variables for the Korteweg-de Vries equation and the Toda lattice with periodic boundary conditions*. Progr. Theoret. Phys. **55**(2), pp. 438–456.
- [9] L. Haine and E. Horozov (1987). *A Lax pair for Kowalevski’s top*. Phys. D **29**(1-2), pp. 173–180.
- [10] E. Horozov and P. van Moerbeke (1989). *The full geometry of Kowalevski’s top and $(1, 2)$ -abelian surfaces*. Comm. Pure Appl. Math. **42**(4), pp. 357–407.
- [11] J. C. Hurtubise (1997). *The algebraic geometry of the Kostant-Kirillov form*. J. London Math. Soc. (2) **56**(3), pp. 504–518.
- [12] J. C. Hurtubise and M. Kjiri (2000). *Separating coordinates for the generalized Hitchin systems and the classical r -matrices*. Comm. Math. Phys. **210**(2), pp. 521–540.
- [13] E. G. Kalnins (1986). *Separation of variables for Riemannian spaces of constant curvature*. Harlow: Longman Scientific & Technical.
- [14] I. V. Komarov (1981). *The Kovalevskaja basis for the hydrogen atom*. Teoret. Mat. Fiz. **47**(1), pp. 67–72.
- [15] I. V. Komarov (1982a). *The Goryachev-Chaplygin top in quantum mechanics*. Teoret. Mat. Fiz. **50**(3), pp. 402–409.
- [16] I. V. Komarov (1982b). *Exact solution of the Goryachev-Chaplygin problem in quantum mechanics*. J. Phys. A **15**(6), pp. 1765–1773.
- [17] I. V. Komarov (2001). *Remarks on Kowalevski’s top*. J. Phys. A **34**(11), pp. 2111–2120. Kowalevski Workshop on Mathematical Methods of Regular Dynamics (Leeds, 2000).
- [18] I. V. Komarov and V. B. Kuznetsov (1987). *Quasiclassical quantization of the Kovalevskaya top*. Teoret. Mat. Fiz. **73**(3), pp. 335–347; English transl. (1987) Theor. Math. Phys. **73**, pp. 1255–1263.
- [19] I. V. Komarov and V. B. Kuznetsov (1990). *Kowalevski’s top on the Lie algebras $\mathfrak{o}(4)$, $\mathfrak{e}(3)$ and $\mathfrak{o}(3, 1)$* . J. Phys. A **23**(6), pp. 841–846.
- [20] T. H. Koornwinder and V. B. Kuznetsov (1994). *Gauss hypergeometric function and quadratic R -matrix algebras*. Algebra i Analiz **6**(3), pp. 161–184.

- [21] Sophie Kowalevski (1889). *Sur le problème de la rotation d'un corps solide autour d'un point fixe*, Acta Math. **12**, pp. 177–232.
- [22] V. B. Kuznetsov (1985). *Quasiclassical quantization of the Kowalevski top*, M.Sc. Thesis, Leningrad University.
- [23] V. B. Kuznetsov (1989). *Applications of the inverse scattering method to 2-dimensional classical and quantum integrable systems*, PhD Thesis, Leningrad University, 118 pp.
- [24] V. B. Kuznetsov (1990). *Generalized polyspheroidal periodic functions and the quantum inverse scattering method*. J. Math. Phys. **31**(5), pp. 1167–1174.
- [25] V. B. Kuznetsov (1996). ${}_3F_2(1)$ hypergeometric function and quadratic R -matrix algebra. In *Symmetries and integrability of difference equations (Estérel, PQ, 1994)*, pp. 185–197. Providence, RI: Amer. Math. Soc.
- [26] V. B. Kuznetsov (1997). *Separation of variables for the D_n -type periodic Toda lattice*. J. Phys. A **30**(6), pp. 2127–2138.
- [27] V. B. Kuznetsov, M. F. Jørgensen, and P. L. Christiansen (1995). *New boundary conditions for integrable lattices*. J. Phys. A **28**(16), pp. 4639–4654.
- [28] V. B. Kuznetsov and A. V. Tsiganov (1989a). *A special case of Neumann's system and the Kowalewski-Chaplygin-Goryachev top*. J. Phys. A **22**(3), pp. L73–L79.
- [29] V. B. Kuznetsov and A. V. Tsiganov (1989b). *Infinite series of Lie algebras and boundary conditions for integrable systems*. Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **172**(Differentsialnaya Geom. Gruppy Li i Mekh. Vol. 10), pp. 88–98, 170.
- [30] V. B. Kuznetsov and A. V. Tsiganov (1993). *Quantum relativistic Toda lattices*. Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **205**(Differentsialnaya Geom. Gruppy Li i Mekh. 13), pp. 71–84, 180.
- [31] D. Markushevich (2001). *Kowalevski top and genus-2 curves*. J. Phys. A **34**(11), pp. 2125–2135. Kowalevski Workshop on Mathematical Methods of Regular Dynamics (Leeds, 2000).
- [32] P. van Moerbeke (1976). *The spectrum of Jacobi matrices*. Invent. Math. **37**(1), pp. 45–81.
- [33] D. Mumford (1984). *Tata lectures on theta. II*. Boston, MA: Birkhäuser Boston Inc. Jacobian theta functions and differential equations, With the collaboration of C. Musili, M. Nori, E. Previato, M. Stillman and H. Umemura.
- [34] E. Previato (1987). *Flows on r -gonal Jacobians*. In *The legacy of Sonya Kovalevskaya (Cambridge, Mass., and Amherst, Mass., 1985)*, pp. 153–180. Providence, RI: Amer. Math. Soc.

- [35] A. G. Reyman and M. A. Semenov-Tian-Shansky (1987). *Lax representation with a spectral parameter for the Kowalewski top and its generalizations*. Lett. Math. Phys. **14**(1), pp. 55–61.
- [36] E. K. Sklyanin (1984). *The Goryachev-Chaplygin top and the method of the inverse scattering problem*. Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **133**, pp. 236–257. Differential geometry, Lie groups and mechanics, VI.
- [37] E. K. Sklyanin (1985). *The quantum Toda chain*. In *Nonlinear equations in classical and quantum field theory (Meudon/Paris, 1983/1984)*, pp. 196–233. Berlin: Springer.
- [38] E. K. Sklyanin (1988). *Boundary conditions for integrable quantum systems*. J. Phys. A **21**(10), pp. 2375–2389.
- [39] E. K. Sklyanin (1995). *Separation of variables—new trends*. Progr. Theoret. Phys. Suppl. (118), pp. 35–60. Quantum field theory, integrable models and beyond (Kyoto, 1994).
- [40] P. Vanhaecke (1996). *Integrable systems in the realm of algebraic geometry*. Berlin: Springer-Verlag.
- [41] A. P. Veselov and S. P. Novikov (1984). *Poisson brackets and complex tori*. Trudy Mat. Inst. Steklov. **165**, pp. 49–61. Algebraic geometry and its applications.